Chapter 3

TRIGONOMETRIC FUNCTIONS

3.1 Overview

3.1.1 The word 'trigonometry' is derived from the Greek words 'trigon' and 'metron' which means measuring the sides of a triangle. An angle is the amount of rotation of a revolving line with respect to a fixed line. If the rotation is in clockwise direction the angle is negative and it is positive if the rotation is in the anti-clockwise direction. Usually we follow two types of conventions for measuring angles, i.e., (i) Sexagesimal system (ii) Circular system.

In sexagesimal system, the unit of measurement is degree. If the rotation from the initial to terminal side is $\frac{1}{360}$th of a revolution, the angle is said to have a measure of 1°. The classifications in this system are as follows:

$$1° = 60'$$

$$1' = 60''$$

In circular system of measurement, the unit of measurement is radian. One radian is the angle subtended, at the centre of a circle, by an arc equal in length to the radius of the circle. The length s of an arc PQ of a circle of radius r is given by $s = r\theta$, where θ is the angle subtended by the arc PQ at the centre of the circle measured in terms of radians.

3.1.2 *Relation between degree and radian*

The circumference of a circle always bears a constant ratio to its diameter. This constant ratio is a number denoted by π which is taken approximately as $\frac{22}{7}$ for all practical purpose. The relationship between degree and radian measurements is as follows:

$$2 \text{ right angle} = 180° = \pi \text{ radians}$$

$$1 \text{ radian} = \frac{180°}{\pi} = 57°16' \text{ (approx)}$$

$$1° = \frac{\pi}{180} \text{ radian} = 0.01746 \text{ radians (approx)}$$

3.1.3 *Trigonometric functions*

Trigonometric ratios are defined for acute angles as the ratio of the sides of a right angled triangle. The extension of trigonometric ratios to any angle in terms of radian measure (real numbers) are called trigonometric functions. The signs of trigonometric functions in different quadrants have been given in the following table:

	I	II	III	IV
$\sin x$	+	+	–	–
$\cos x$	+	–	–	+
$\tan x$	+	–	+	–
$\operatorname{cosec} x$	+	+	–	–
$\sec x$	+	–	–	+
$\cot x$	+	–	+	–

3.1.4 *Domain and range of trigonometric functions*

Functions	**Domain**	**Range**
sine	$\mathbf{R}$	$[-1, 1]$
cosine	$\mathbf{R}$	$[-1, 1]$
tan	$\mathbf{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\}$	$\mathbf{R}$
cot	$\mathbf{R} - \{n\pi : n \in \mathbf{Z}\}$	$\mathbf{R}$
sec	$\mathbf{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\}$	$\mathbf{R} - (-1, 1)$
cosec	$\mathbf{R} - \{n\pi : n \in \mathbf{Z}\}$	$\mathbf{R} - (-1, 1)$

3.1.5 *Sine, cosine and tangent of some angles less than 90°*

	0°	15°	18°	30°	36°	45°	60°	90°
sine	0	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

cosine	1	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$2-\sqrt{3}$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\frac{1}{\sqrt{3}}$	$\sqrt{5-2\sqrt{5}}$	1	$\sqrt{3}$	not defined

3.1.6 *Allied or related angles* The angles $\frac{n\pi}{2} \pm \theta$ are called allied or related angles and $\theta \pm n \times 360°$ are called coterminal angles. For general reduction, we have the following rules. The value of any trigonometric function for $(\frac{n\pi}{2} \pm \theta)$ is numerically equal to

(a) the value of the same function if n is an even integer with algebaric sign of the function as per the quadrant in which angles lie.

(b) corresponding cofunction of θ if n is an odd integer with algebraic sign of the function for the quadrant in which it lies. Here sine and cosine; tan and cot; sec and cosec are cofunctions of each other.

3.1.7 *Functions of negative angles* Let θ be any angle. Then

$$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta, \quad \cot(-\theta) = -\cot\theta$$

$$\sec(-\theta) = \sec\theta, \quad \operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$$

3.1.8 *Some formulae regarding compound angles*

An angle made up of the sum or differences of two or more angles is called a compound angle. The basic results in this direction are called trigonometric identies as given below:

(i) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(ii) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

(iii) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

(iv) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

(vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

(vii) $\cot(A + B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$

(viii) $\cot(A - B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

(ix) $\sin 2A = 2 \sin A \cos A = \dfrac{2 \tan A}{1 + \tan^2 A}$

(x) $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A = 2 \cos^2 A - 1 = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

(xi) $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

(xii) $\sin 3A = 3\sin A - 4\sin^3 A$

(xiii) $\cos 3A = 4\cos^3 A - 3\cos A$

(xiv) $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3\tan^2 A}$

(xv) $\cos A + \cos B = 2 \cos \dfrac{A + B}{2} \cos \dfrac{A - B}{2}$

(xvi) $\cos A - \cos B = 2\sin \dfrac{A + B}{2} \sin \dfrac{B - A}{2}$

(xvii) $\sin A + \sin B = 2\sin \dfrac{A + B}{2} \cos \dfrac{A - B}{2}$

(xviii) $\sin A - \sin B = 2\cos \dfrac{A + B}{2} \sin \dfrac{A - B}{2}$

(xix) $2\sin A \cos B = \sin(A + B) + \sin(A - B)$

(xx) $2\cos A \sin B = \sin(A + B) - \sin(A - B)$

(xxi) $2\cos A \cos B = \cos(A + B) + \cos(A - B)$

(xxii) $2\sin A \sin B = \cos(A - B) - \cos(A + B)$

(xxiii) $\sin \dfrac{A}{2} = \pm \sqrt{\dfrac{1 - \cos A}{2}}$ $+$ if $\dfrac{A}{2}$ lies in quadrants I or II; $-$ if $\dfrac{A}{2}$ lies in III or IV quadrants

(xxiv) $$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}} \quad \begin{cases} + \text{ if } \frac{A}{2} \text{ lies in I or IV quadrants} \\ - \text{ if } \frac{A}{2} \text{ lies in II or III quadrants} \end{cases}$$

(xxv) $$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} \quad \begin{cases} + \text{ if } \frac{A}{2} \text{ lies in I or III quadrants} \\ - \text{ if } \frac{A}{2} \text{ lies in II or IV quadrants} \end{cases}$$

Trigonometric functions of an angle of 18°

Let $\theta = 18°$. Then $2\theta = 90° - 3\theta$

Therefore, $\sin 2\theta = \sin (90° - 3\theta) = \cos 3\theta$

or $\sin 2\theta = 4\cos^3\theta - 3\cos\theta$

Since, $\cos\theta \neq 0$, we get

$2\sin\theta = 4\cos^2\theta - 3 = 1 - 4\sin^2\theta$ or $4\sin^2\theta + 2\sin\theta - 1 = 0$.

Hence, $$\sin\theta = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-1\pm\sqrt{5}}{4}$$

Since, $\theta = 18°$, $\sin\theta > 0$, therefore, $\sin 18° = \frac{\sqrt{5}-1}{4}$

Also, $$\cos 18° = \sqrt{1-\sin^2 18°} = \sqrt{1-\frac{6-2\sqrt{5}}{16}} = \sqrt{\frac{10+2\sqrt{5}}{4}}$$

Now, we can easily find cos 36° and sin 36° as follows:

$$\cos 36° = 1 - 2\sin^2 18° = 1 - \frac{6-2\sqrt{5}}{8} = \frac{2+2\sqrt{5}}{8} = \frac{\sqrt{5}+1}{4}$$

Hence, $$\cos 36° = \frac{\sqrt{5}+1}{4}$$

Also, $$\sin 36° = \sqrt{1-\cos^2 36°} = \sqrt{1-\frac{6+2\sqrt{5}}{16}} = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

3.1.9 *Trigonometric equations*

Equations involving trigonometric functions of a variables are called **trigonometric equations**. Equations are called identities, if they are satisfied by all values of the unknown angles for which the functions are defined. The solutions of a trigonometric equations for which $0 \le \theta < 2\pi$ are called **principal solutions**. The expression involving integer n which gives all solutions of a trigonometric equation is called the **general solution**.

General Solution of Trigonometric Equations

(i) If $\sin\theta = \sin\alpha$ for some angle α, then
$\theta = n\pi + (-1)^n\alpha$ for $n \in \mathbf{Z}$, gives general solution of the given equation

(ii) If $\cos\theta = \cos\alpha$ for some angle α, then
$\theta = 2n\pi \pm \alpha$, $n \in \mathbf{Z}$, gives general solution of the given equation

(iii) If $\tan\theta = \tan\alpha$ or $\cot\theta = \cot\alpha$, then
$\theta = n\pi + \alpha$, $n \in \mathbf{Z}$, gives general solution for both equations

(iv) The general value of θ satisfying any of the equations $\sin^2\theta = \sin^2\alpha$, $\cos^2\theta = \cos^2\alpha$ and
$\tan^2\theta = \tan^2\alpha$ is given by $\theta = n\pi \pm \alpha$

(v) The general value of θ satisfying equations $\sin\theta = \sin\alpha$ and $\cos\theta = \cos\alpha$ simultaneously is given by $\theta = 2n\pi + \alpha$, $n \in \mathbf{Z}$.

(vi) To find the solution of an equation of the form $a\cos\theta + b\sin\theta = c$, we put

$a = r\cos\alpha$ and $b = r\sin\alpha$, so that $r^2 = a^2 + b^2$ and $\tan\alpha = \dfrac{b}{a}$.

Thus we find

$a\cos\theta + b\sin\theta = c$ changed into the form $r(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = c$

or $r\cos(\theta - \alpha) = c$ and hence $\cos(\theta - \alpha) = \dfrac{c}{r}$. This gives the solution of the given equation.

Maximum and Minimum values of the expression $A\cos\theta + B\sin\theta$ are $\sqrt{A^2 + B^2}$ and $-\sqrt{A^2 + B^2}$ respectively, where A and B are constants.

3.2 Solved Examples

Short Answer Type

Example 1 A circular wire of radius 3 cm is cut and bent so as to lie along the circumference of a hoop whose radius is 48 cm. Find the angle in degrees which is subtended at the centre of hoop.

Solution Given that circular wire is of radius 3 cm, so when it is cut then its length $= 2\pi \times 3 = 6\pi$ cm. Again, it is being placed along a circular hoop of radius 48 cm. Here, $s = 6\pi$ cm is the length of arc and $r = 48$ cm is the radius of the circle. Therefore, the angle θ, in radian, subtended by the arc at the centre of the circle is given by

$$\theta = \frac{\text{Arc}}{\text{Radius}} = \frac{6\pi}{48} = \frac{\pi}{8} = 22.5^\circ.$$

Example 2 If $A = \cos^2\theta + \sin^4\theta$ for all values of θ, then prove that $\frac{3}{4} \le A \le 1$.

Solution We have $A = \cos^2\theta + \sin^4\theta = \cos^2\theta + \sin^2\theta \sin^2\theta \le \cos^2\theta + \sin^2\theta$

Therefore, $A \le 1$

Also, $A = \cos^2\theta + \sin^4\theta = (1 - \sin^2\theta) + \sin^4\theta$

$$= \left(\sin^2\theta - \frac{1}{2}\right)^2 + \left(1 - \frac{1}{4}\right) = \left(\sin^2\theta - \frac{1}{2}\right)^2 + \frac{3}{4} \ge \frac{3}{4}$$

Hence, $\frac{3}{4} \le A \le 1$.

Example 3 Find the value of $\sqrt{3}$ cosec 20° – sec 20°

Solution We have

$$\sqrt{3}\ \text{cosec}\ 20^\circ - \sec 20^\circ = \frac{\sqrt{3}}{\sin 20^\circ} - \frac{1}{\cos 20^\circ}$$

$$= \frac{\sqrt{3}\cos 20^\circ - \sin 20^\circ}{\sin 20^\circ \cos 20^\circ} = 4\left(\frac{\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ}{2\sin 20^\circ \cos 20^\circ}\right)$$

$$= 4\left(\frac{\sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ}{\sin 40^\circ}\right)$$

(Why?)

$$= 4\left(\frac{\sin(60^\circ - 20^\circ)}{\sin 40^\circ}\right) = 4 \qquad \text{(Why?)}$$

Example 4 If θ lies in the second quadrant, then show that

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}+\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}=-2\sec\theta$$

Solution We have

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}+\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}=\frac{1-\sin\theta}{\sqrt{1-\sin^2\theta}}+\frac{1+\sin\theta}{\sqrt{1-\sin^2\theta}}=\frac{2}{\sqrt{\cos^2\theta}}$$

$$=\frac{2}{|\cos\theta|} \qquad \left(\text{Since } \sqrt{\alpha^2}=|\alpha| \text{ for every real number } \alpha\right)$$

Given that θ lies in the second quadrant so $|\cos\theta| = -\cos\theta$ (since $\cos\theta < 0$).

Hence, the required value of the expression is $\dfrac{2}{-\cos\theta}=-2\sec\theta$

Example 5 Find the value of tan 9° – tan 27° – tan 63° + tan 81°

Solution We have tan 9° – tan 27° – tan 63° + tan 81°

$$= \tan 9° + \tan 81° - \tan 27° - \tan 63°$$

$$= \tan 9° + \tan (90° - 9°) - \tan 27° - \tan (90° - 27°)$$

$$= \tan 9° + \cot 9° - (\tan 27° + \cot 27°) \qquad (1)$$

Also $\quad \tan 9° + \cot 9° = \dfrac{1}{\sin 9° \cos 9°}=\dfrac{2}{\sin 18°}$ (Why?) (2)

Similarly, $\quad \tan 27° + \cot 27° = \dfrac{1}{\sin 27° \cos 27°}=\dfrac{2}{\sin 54°}=\dfrac{2}{\cos 36°}$ (Why?) (3)

Using (2) and (3) in (1), we get

$$\tan 9° - \tan 27° - \tan 63° + \tan 81° = \frac{2}{\sin 18°}-\frac{2}{\cos 36°}=\frac{2\times 4}{\sqrt{5}-1}-\frac{2\times 4}{\sqrt{5}+1}=4$$

Example 6 Prove that $\dfrac{\sec 8\theta-1}{\sec 4\theta-1}=\dfrac{\tan 8\theta}{\tan 2\theta}$

Solution We have $\quad \dfrac{\sec 8\theta-1}{\sec 4\theta-1}=\dfrac{(1-\cos 8\theta)\cos 4\theta}{\cos 8\theta\,(1-\cos 4\theta)}$

$$=\frac{2\sin^2 4\theta\cos 4\theta}{\cos 8\theta\, 2\sin^2 2\theta} \qquad \text{(Why?)}$$

$$= \frac{\sin 4\theta\,(2\sin 4\theta \cos 4\theta)}{2\cos 8\theta \sin^2 2\theta}$$

$$= \frac{\sin 4\theta \sin 8\theta}{2\cos 8\theta \sin^2 2\theta} \qquad \text{(Why?)}$$

$$= \frac{2\sin 2\theta \cos 2\theta \sin 8\theta}{2\cos 8\theta \sin^2 2\theta}$$

$$= \frac{\tan 8\theta}{\tan 2\theta} \qquad \text{(Why?)}$$

Example 7 Solve the equation $\sin\theta + \sin 3\theta + \sin 5\theta = 0$

Solution We have $\sin\theta + \sin 3\theta + \sin 5\theta = 0$

or $(\sin\theta + \sin 5\theta) + \sin 3\theta = 0$

or $2\sin 3\theta \cos 2\theta + \sin 3\theta = 0$ (Why?)

or $\sin 3\theta\,(2\cos 2\theta + 1) = 0$

or $\sin 3\theta = 0$ or $\cos 2\theta = -\frac{1}{2}$

When $\sin 3\theta = 0$, then $3\theta = n\pi$ or $\theta = \frac{n\pi}{3}$

When $\cos 2\theta = -\frac{1}{2} = \cos\frac{2\pi}{3}$, then $2\theta = 2n\pi \pm \frac{2\pi}{3}$ or $\theta = n\pi \pm \frac{\pi}{3}$

which gives $\theta = (3n+1)\frac{\pi}{3}$ or $\theta = (3n-1)\frac{\pi}{3}$

All these values of θ are contained in $\theta = \frac{n\pi}{3}, n \in \mathbf{Z}$. Hence, the required solution set is given by $\{\theta : \theta = \frac{n\pi}{3}, n \in \mathbf{Z}\}$

Example 8 Solve $2\tan^2 x + \sec^2 x = 2$ for $0 \le x \le 2\pi$

Solution Here, $2\tan^2 x + \sec^2 x = 2$

which gives $\tan x = \pm\frac{1}{\sqrt{3}}$

If we take $\tan x = \dfrac{1}{\sqrt{3}}$, then $x = \dfrac{\pi}{6}$ or $\dfrac{7\pi}{6}$ (Why?)

Again, if we take $\tan x = \dfrac{-1}{\sqrt{3}}$, then $x = \dfrac{5\pi}{6}$ or $\dfrac{11\pi}{6}$ (Why?)

Therefore, the possible solutions of above equations are

$x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}$ and $\dfrac{11\pi}{6}$ where $0 \le x \le 2\pi$

Long Answer Type

Example 9 Find the value of $\left(1+\cos\dfrac{\pi}{8}\right)\left(1+\cos\dfrac{3\pi}{8}\right)\left(1+\cos\dfrac{5\pi}{8}\right)\left(1+\cos\dfrac{7\pi}{8}\right)$

Solution Write $\left(1+\cos\dfrac{\pi}{8}\right)\left(1+\cos\dfrac{3\pi}{8}\right)\left(1+\cos\dfrac{5\pi}{8}\right)\left(1+\cos\dfrac{7\pi}{8}\right)$

$$= \left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\left(\pi-\frac{3\pi}{8}\right)\right)\left(1+\cos\left(\pi-\frac{\pi}{8}\right)\right)$$

$$= \left(1-\cos^2\frac{\pi}{8}\right)\left(1-\cos^2\frac{3\pi}{8}\right) \qquad \text{(Why?)}$$

$$= \sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}$$

$$= \frac{1}{4}\left(1-\cos\frac{\pi}{4}\right)\left(1-\cos\frac{3\pi}{4}\right) \qquad \text{(Why?)}$$

$$= \frac{1}{4}\left(1-\cos\frac{\pi}{4}\right)\left(1+\cos\frac{\pi}{4}\right) \qquad \text{(Why?)}$$

$$= \frac{1}{4}\left(1-\cos^2\frac{\pi}{4}\right) = \frac{1}{4}\left(1-\frac{1}{2}\right) = \frac{1}{8}$$

Example 10 If $x\cos\theta = y\cos\left(\theta + \dfrac{2\pi}{3}\right) = z\cos\left(\theta + \dfrac{4\pi}{3}\right)$, then find the value of $xy + yz + zx$.

Solution Note that $xy + yz + zx = xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$.

If we put $x \cos\theta = y \cos(\theta + \frac{2\pi}{3}) = z \cos\left(\theta+\frac{4\pi}{3}\right) = k$ (say).

Then $$x = \frac{k}{\cos\theta},\ y = \frac{k}{\cos\left(\theta+\frac{2\pi}{3}\right)} \text{ and } z = \frac{k}{\cos\left(\theta+\frac{4\pi}{3}\right)}$$

so that $$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{1}{k}\left[\cos\theta+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{4\pi}{3}\right)\right]$$

$$= \frac{1}{k}[\cos\theta+\cos\theta\cos\frac{2\pi}{3}-\sin\theta\sin\frac{2\pi}{3}$$

$$+ \cos\theta\cos\frac{4\pi}{3}-\sin\theta\sin\frac{4\pi}{3}]$$

$$= \frac{1}{k}[\cos\theta+\cos\theta\,(\frac{-1}{2})-\frac{\sqrt{3}}{2}\sin\theta-\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta] \text{ (Why?)}$$

$$= \frac{1}{k}\times 0 = 0$$

Hence, $xy + yz + zx = 0$

Example 11 If α and β are the solutions of the equation $a \tan\theta + b \sec\theta = c$, then show that $\tan(\alpha + \beta) = \frac{2ac}{a^2-c^2}$.

Solution Given that $a\tan\theta + b\sec\theta = c$ or $a\sin\theta + b = c\cos\theta$

Using the identities,

$$\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \text{ and } \cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

We have, $$\frac{a\left(2\tan\frac{\theta}{2}\right)}{1+\tan^2\frac{\theta}{2}}+b=\frac{c\left(1-\tan^2\frac{\theta}{2}\right)}{1+\tan^2\frac{\theta}{2}}$$

or $$(b + c)\ \tan^2\frac{\theta}{2} + 2a\ \tan\frac{\theta}{2} + b - c = 0$$

Above equation is quadratic in $\tan\frac{\theta}{2}$ and hence $\tan\frac{\alpha}{2}$ and $\tan\frac{\beta}{2}$ are the roots of this equation (Why?). Therefore, $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} = \frac{-2a}{b+c}$ and $\tan\frac{\alpha}{2}\ \tan\frac{\beta}{2} = \frac{b-c}{b+c}$ (Why?)

Using the identity $$\tan\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)=\frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$$

We have, $$\tan\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)=\frac{\frac{-2a}{b+c}}{1-\frac{b-c}{b+c}} = \frac{-2a}{2c}=\frac{-a}{c} \qquad \text{... (1)}$$

Again, using another identity

$$\tan 2\ \frac{\alpha+\beta}{2} = \frac{2\tan\frac{\alpha+\beta}{2}}{1-\tan^2\frac{\alpha+\beta}{2}},$$

We have $$\tan(\alpha+\beta)=\frac{2\left(-\frac{a}{c}\right)}{1-\frac{a^2}{c^2}} = \frac{2ac}{a^2-c^2} \quad \text{[From (1)]}$$

Alternatively, given that $a\tan\theta + b\sec\theta = c$

$\Rightarrow \quad (a \tan\theta - c)^2 = b^2(1 + \tan^2\theta)$

$\Rightarrow \quad a^2 \tan^2\theta - 2ac \tan\theta + c^2 = b^2 + b^2 \tan^2\theta$

$\Rightarrow \quad (a^2 - b^2) \tan^2\theta - 2ac \tan\theta + c^2 - b^2 = 0 \qquad \text{... (1)}$

Since α and β are the roots of the equation (1), so

$$\tan\alpha + \tan\beta = \frac{2ac}{a^2 - b^2} \quad \text{and} \quad \tan\alpha \tan\beta = \frac{c^2 - b^2}{a^2 - b^2}$$

Therefore,
$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$$

$$= \frac{\dfrac{2ac}{a^2 - b^2}}{\dfrac{c^2 - b^2}{a^2 - b^2}} = \frac{2ac}{a^2 - c^2}$$

Example 12 Show that $2 \sin^2\beta + 4 \cos(\alpha + \beta) \sin\alpha \sin\beta + \cos 2(\alpha + \beta) = \cos 2\alpha$

Solution LHS $= 2 \sin^2\beta + 4 \cos(\alpha + \beta) \sin\alpha \sin\beta + \cos 2(\alpha + \beta)$

$= 2 \sin^2\beta + 4 (\cos\alpha \cos\beta - \sin\alpha \sin\beta) \sin\alpha \sin\beta$
$\quad + (\cos 2\alpha \cos 2\beta - \sin 2\alpha \sin 2\beta)$

$= 2 \sin^2\beta + 4 \sin\alpha \cos\alpha \sin\beta \cos\beta - 4 \sin^2\alpha \sin^2\beta$
$\quad + \cos 2\alpha \cos 2\beta - \sin 2\alpha \sin 2\beta$

$= 2 \sin^2\beta + \sin 2\alpha \sin 2\beta - 4 \sin^2\alpha \sin^2\beta + \cos 2\alpha \cos 2\beta - \sin 2\alpha \sin 2\beta$

$= (1 - \cos 2\beta) - (2 \sin^2\alpha)(2 \sin^2\beta) + \cos 2\alpha \cos 2\beta$ (Why?)

$= (1 - \cos 2\beta) - (1 - \cos 2\alpha)(1 - \cos 2\beta) + \cos 2\alpha \cos 2\beta$

$= \cos 2\alpha$ (Why?)

Example 13 If angle θ is divided into two parts such that the tangent of one part is k times the tangent of other, and ϕ is their difference, then show that

$$\sin\theta = \frac{k+1}{k-1} \sin\phi$$

Solution Let $\theta = \alpha + \beta$. Then $\tan\alpha = k \tan\beta$

or $$\frac{\tan\alpha}{\tan\beta} = \frac{k}{1}$$

Applying componendo and dividendo, we have

$$\frac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta} = \frac{k+1}{k-1}$$

or $$\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\cos\beta - \cos\alpha\sin\beta} = \frac{k+1}{k-1}$$ (Why?)

i.e., $$\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{k+1}{k-1}$$ (Why?)

Given that $\alpha - \beta = \phi$ and $\alpha + \beta = \theta$. Therefore,

$$\frac{\sin\theta}{\sin\phi} = \frac{k+1}{k-1} \quad \text{or} \quad \sin\theta = \frac{k+1}{k-1}\sin\phi$$

Example 14 Solve $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$

Solution Divide the given equation by 2 to get

$$\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta = \frac{1}{\sqrt{2}} \quad \text{or} \quad \cos\frac{\pi}{6}\cos\theta + \sin\frac{\pi}{6}\sin\theta = \cos\frac{\pi}{4}$$

or $$\cos\left(\frac{\pi}{6} - \theta\right) = \cos\frac{\pi}{4} \quad \text{or} \quad \cos\left(\theta - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}$$ (Why?)

Thus, the solution are given by, i.e., $\theta = 2m\pi \pm \frac{\pi}{4} + \frac{\pi}{6}$

Hence, the solution are

$$\theta = 2m\pi + \frac{\pi}{4} + \frac{\pi}{6} \quad \text{and} \quad 2m\pi - \frac{\pi}{4} + \frac{\pi}{6}, \quad \text{i.e.,} \quad \theta = 2m\pi + \frac{5\pi}{12} \quad \text{and} \quad \theta = 2m\pi - \frac{\pi}{12}$$

Objective Type Questions

Choose the correct answer from the given four options against each of the Examples 15 to 19

Example 15 If $\tan\theta = \frac{-4}{3}$, then $\sin\theta$ is

(A) $\frac{-4}{5}$ but not $\frac{4}{5}$ (B) $\frac{-4}{5}$ or $\frac{4}{5}$

(C) $\frac{4}{5}$ but not $-\frac{4}{5}$ (D) None of these

Solution Correct choice is B. Since $\tan\theta = -\frac{4}{3}$ is negative, θ lies either in second quadrant or in fourth quadrant. Thus $\sin\theta = \frac{4}{5}$ if θ lies in the second quadrant or $\sin\theta = -\frac{4}{5}$, if θ lies in the fourth quadrant.

Example 16 If sin θ and cos θ are the roots of the equation $ax^2 - bx + c = 0$, then a, b and c satisfy the relation.

(A) $a^2 + b^2 + 2ac = 0$ (B) $a^2 - b^2 + 2ac = 0$

(C) $a^2 + c^2 + 2ab = 0$ (D) $a^2 - b^2 - 2ac = 0$

Solution The correct choice is (B). Given that sin θ and cos θ are the roots of the equation $ax^2 - bx + c = 0$, so $\sin\theta + \cos\theta = \frac{b}{a}$ and $\sin\theta\cos\theta = \frac{c}{a}$ (Why?)

Using the identity $(\sin\theta + \cos\theta)^2 = \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta$, we have

$$\frac{b^2}{a^2} = 1 + \frac{2c}{a} \text{ or } a^2 - b^2 + 2ac = 0$$

Example 17 The greatest value of $\sin x \cos x$ is

(A) 1 (B) 2 (C) $\sqrt{2}$ (D) $\frac{1}{2}$

Solution (D) is the correct choice, since

$$\sin x \ \cos x = \frac{1}{2}\sin 2x \le \frac{1}{2}, \text{ since } |\sin 2x| \le 1.$$

Eaxmple 18 The value of sin 20° sin 40° sin 60° sin 80° is

(A) $\frac{-3}{16}$ (B) $\frac{5}{16}$ (C) $\frac{3}{16}$ (D) $\frac{1}{16}$

Solution Correct choice is (C). Indeed sin 20° sin 40° sin 60° sin 80°.

$$= \frac{\sqrt{3}}{2} \sin 20° \sin (60° - 20°) \sin (60° + 20°) \text{ (since } \sin 60° = \frac{\sqrt{3}}{2}\text{)}$$

$$= \frac{\sqrt{3}}{2} \sin 20° [\sin^2 60° - \sin^2 20°] \qquad \text{(Why?)}$$

$$= \frac{\sqrt{3}}{2} \sin 20° [\frac{3}{4} - \sin^2 20°]$$

$$= \frac{\sqrt{3}}{2} \times \frac{1}{4} [3\sin 20° - 4\sin^3 20°]$$

$$= \frac{\sqrt{3}}{2} \times \frac{1}{4} (\sin 60°) \qquad \text{(Why?)}$$

$$= \frac{\sqrt{3}}{2} \times \frac{1}{4} \times \frac{\sqrt{3}}{2} = \frac{3}{16}$$

Example 19 The value of $\cos \frac{\pi}{5} \cos \frac{2\pi}{5} \cos \frac{4\pi}{5} \cos \frac{8\pi}{5}$ is

(A) $\frac{1}{16}$ (B) 0 (C) $\frac{-1}{8}$ (D) $\frac{-1}{16}$

Solution (D) is the correct answer. We have

$$\cos \frac{\pi}{5} \cos \frac{2\pi}{5} \cos \frac{4\pi}{5} \cos \frac{8\pi}{5}$$

$$= \frac{1}{2\sin\frac{\pi}{5}} 2\sin\frac{\pi}{5} \cos\frac{\pi}{5} \cos\frac{2\pi}{5} \cos\frac{4\pi}{5} \cos\frac{8\pi}{5}$$

$$= \frac{1}{2\sin\frac{\pi}{5}} \sin\frac{2\pi}{5} \cos\frac{2\pi}{5} \cos\frac{4\pi}{5} \cos\frac{8\pi}{5} \qquad \text{(Why?)}$$

$$= \frac{1}{4\sin\frac{\pi}{5}} \sin\frac{4\pi}{5} \cos\frac{4\pi}{5} \cos\frac{8\pi}{5} \qquad \text{(Why?)}$$

$$= \frac{1}{8\sin\frac{\pi}{5}} \sin\frac{8\pi}{5} \cos\frac{8\pi}{5} \qquad \text{(Why?)}$$

$$= \frac{\sin\frac{16\pi}{5}}{16\sin\frac{\pi}{5}} = \frac{\sin\left(3\pi+\frac{\pi}{5}\right)}{16\sin\frac{\pi}{5}}$$

$$= \frac{-\sin\frac{\pi}{5}}{16\sin\frac{\pi}{5}} \qquad \text{(Why?)}$$

$$= -\frac{1}{16}$$

Fill in the blank :

Example 20 If 3 tan (θ – 15°) = tan (θ + 15°), 0° < θ < 90°, then θ = __________

Solution Given that 3 tan (θ – 15°) = tan (θ + 15°) which can be rewritten as

$$\frac{\tan(\theta+15°)}{\tan(\theta-15°)} = \frac{3}{1}.$$

Applying componendo and Dividendo; we get $\frac{\tan(\theta+15°)+\tan(\theta-15°)}{\tan(\theta+15°)-\tan(\theta-15°)} = 2$

$$\Rightarrow \frac{\sin(\theta+15°)\cos(\theta-15°)+\sin(\theta-15°)\cos(\theta+15°)}{\sin(\theta+15°)\cos(\theta-15°)-\sin(\theta-15°)\cos(\theta+15°)} = 2$$

$$\Rightarrow \frac{\sin 2\theta}{\sin 30°} = 2 \quad \text{i.e.,} \quad \sin 2\theta = 1 \qquad \text{(Why?)}$$

giving $\theta = \frac{\pi}{4}$

State whether the following statement is True or False. Justify your answer

Example 21 "The inequality $2^{\sin\theta} + 2^{\cos\theta} \geq 2^{1-\frac{1}{\sqrt{2}}}$ holds for all real values of θ"

Solution True. Since $2^{\sin\theta}$ and $2^{\cos\theta}$ are positive real numbers, so A.M. (Arithmetic Mean) of these two numbers is greater or equal to their G.M. (Geometric Mean) and hence

$$\frac{2^{\sin\theta}+2^{\cos\theta}}{2} \geq \sqrt{2^{\sin\theta}\times 2^{\cos\theta}} = \sqrt{2^{\sin\theta+\cos\theta}}$$

$$\geq 2^{\frac{\sin\theta+\cos\theta}{2}} = 2^{\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta\right)}$$

$$\geq 2^{\frac{1}{\sqrt{2}}\sin\left(\frac{\pi}{4}+\theta\right)}$$

Since, $\quad -1 \leq \sin\left(\frac{\pi}{4}+\theta\right) \leq 1$, we have

$$\frac{2^{\sin\theta}+2^{\cos\theta}}{2} \geq 2^{\frac{-1}{\sqrt{2}}} \Rightarrow 2^{\sin\theta}+2^{\cos\theta} \geq 2^{1-\frac{1}{\sqrt{2}}}$$

Match each item given under the column C_1 to its correct answer given under column C_2

Example 22

	C_1		C_2
(a)	$\dfrac{1-\cos x}{\sin x}$	(i)	$\cot^2\dfrac{x}{2}$
(b)	$\dfrac{1+\cos x}{1-\cos x}$	(ii)	$\cot\dfrac{x}{2}$
(c)	$\dfrac{1+\cos x}{\sin x}$	(iii)	$\lvert\cos x+\sin x\rvert$
(d)	$\sqrt{1+\sin 2x}$	(iv)	$\tan\dfrac{x}{2}$

Solution

(a) $$\frac{1-\cos x}{\sin x} = \frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \tan\frac{x}{2}.$$

Hence (a) matches with (iv) denoted by (a) $\leftrightarrow$ (iv)

(b) $\dfrac{1+\cos x}{1-\cos x} = \dfrac{2\sin^2\frac{x}{2}}{2\sin^2\frac{x}{2}} = \cot^2\dfrac{x}{2}$. Hence (b) matches with (i) i.e., (b) ↔ (i)

(c) $\dfrac{1+\cos x}{\sin x} = \dfrac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \cot\dfrac{x}{2}$.

Hence (c) matches with (ii) i.e., (c) ↔ (ii)

(d) $\sqrt{1+\sin 2x} = \sqrt{\sin^2 x+\cos^2 x+2\sin x\cos x}$

$= \sqrt{(\sin x+\cos x)^2}$

$= \left|(\sin x+\cos x)\right|$. Hence (d) matches with (iii), i.e., (d) ↔ (iii)

3.3 EXERCISE

Short Answer Type

1. Prove that $\dfrac{\tan A+\sec A-1}{\tan A-\sec A+1} = \dfrac{1+\sin A}{\cos A}$

2. If $\dfrac{2\sin\alpha}{1+\cos\alpha+\sin\alpha} = y$, then prove that $\dfrac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha}$ is also equal to y.

$\left[\textbf{Hint}:\text{Express } \dfrac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha} = \dfrac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha}\cdot\dfrac{1+\cos\alpha+\sin\alpha}{1+\cos\alpha+\sin\alpha}\right]$

3. If $m\sin\theta = n\sin(\theta+2\alpha)$, then prove that $\tan(\theta+\alpha)\cot\alpha = \dfrac{m+n}{m-n}$

[**Hint:** Express $\dfrac{\sin(\theta+2\alpha)}{\sin\theta} = \dfrac{m}{n}$ and apply componendo and dividendo]

4. If $\cos(\alpha+\beta) = \dfrac{4}{5}$ and $\sin(\alpha-\beta) = \dfrac{5}{13}$, where α lie between 0 and $\dfrac{\pi}{4}$, find the value of $\tan 2\alpha$ [**Hint:** Express $\tan 2\alpha$ as $\tan(\alpha+\beta+\alpha-\beta)$]

5. If $\tan x = \dfrac{b}{a}$, then find the value of $\sqrt{\dfrac{a+b}{a-b}} + \sqrt{\dfrac{a-b}{a+b}}$

6. Prove that $\cos\theta \cos\dfrac{\theta}{2} - \cos 3\theta \cos\dfrac{9\theta}{2} = \sin 7\theta \sin 8\theta$.

[**Hint:** Express L.H.S. = $\dfrac{1}{2}[2\cos\theta \cos\dfrac{\theta}{2} - 2\cos 3\theta \cos\dfrac{9\theta}{2}]$]

7. If $a \cos\theta + b \sin\theta = m$ and $a \sin\theta - b\cos\theta = n$, then show that $a^2 + b^2 = m^2 + n^2$

8. Find the value of $\tan 22°30'$.

[**Hint:** Let $\theta = 45°$, use $\tan\dfrac{\theta}{2} = \dfrac{\sin\dfrac{\theta}{2}}{\cos\dfrac{\theta}{2}} = \dfrac{2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}{2\cos^2\dfrac{\theta}{2}} = \dfrac{\sin\theta}{1+\cos\theta}$]

9. Prove that $\sin 4A = 4\sin A \cos^3 A - 4\cos A \sin^3 A$.

10. If $\tan\theta + \sin\theta = m$ and $\tan\theta - \sin\theta = n$, then prove that $m^2 - n^2 = 4\sin\theta \tan\theta$

[**Hint:** $m + n = 2\tan\theta$, $m - n = 2\sin\theta$, then use $m^2 - n^2 = (m + n)(m - n)$]

11. If $\tan(A + B) = p$, $\tan(A - B) = q$, then show that $\tan 2A = \dfrac{p+q}{1-pq}$

[**Hint:** Use $2A = (A + B) + (A - B)$]

12. If $\cos\alpha + \cos\beta = 0 = \sin\alpha + \sin\beta$, then prove that $\cos 2\alpha + \cos 2\beta = -2\cos(\alpha + \beta)$.

[**Hint:** $(\cos\alpha + \cos\beta)^2 - (\sin\alpha + \sin\beta)^2 = 0$]

13. If $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{a+b}{a-b}$, then show that $\dfrac{\tan x}{\tan y} = \dfrac{a}{b}$ [**Hint:** Use Componendo and Dividendo].

14. If $\tan\theta = \dfrac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$, then show that $\sin\alpha + \cos\alpha = \sqrt{2}\cos\theta$.

[**Hint:** Express $\tan\theta = \tan(\alpha - \dfrac{\pi}{4})$ $\quad \theta = \alpha - \dfrac{\pi}{4}$]

15. If $\sin\theta + \cos\theta = 1$, then find the general value of θ.

16. Find the most general value of θ satisfying the equation $\tan\theta = -1$ and $\cos\theta = \dfrac{1}{\sqrt{2}}$.

17. If $\cot\theta + \tan\theta = 2\,\mathrm{cosec}\theta$, then find the general value of θ.

18. If $2\sin^2\theta = 3\cos\theta$, where $0 \le \theta \le 2\pi$, then find the value of θ.

19. If $\sec x \cos 5x + 1 = 0$, where $0 < x \le \frac{\pi}{2}$, then find the value of x.

Long Answer Type

20. If $\sin(\theta + \alpha) = a$ and $\sin(\theta + \beta) = b$, then prove that $\cos 2(\alpha - \beta) - 4ab\cos(\alpha - \beta) = 1 - 2a^2 - 2b^2$ [**Hint:** Express $\cos(\alpha - \beta) = \cos((\theta + \alpha) - (\theta + \beta))$]

21. If $\cos(\theta + \phi) = m\cos(\theta - \phi)$, then prove that $\tan\theta = \frac{1-m}{1+m}\cot\phi$.

[**Hint:** Express $\frac{\cos(\theta+\phi)}{\cos(\theta-\phi)} = \frac{m}{1}$ and apply Componendo and Dividendo]

22. Find the value of the expression

$$3\left[\sin^4\left(\frac{3\pi}{2} - \alpha\right) + \sin^4(3\pi + \alpha)\right] - 2\left\{\sin^6\left(\frac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right]$$

23. If $a\cos 2\theta + b\sin 2\theta = c$ has α and β as its roots, then prove that

$$\tan\alpha + \tan\beta = \frac{2b}{a+c}.$$

[**Hint:** Use the identities $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$ and $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$].

24. If $x = \sec\phi - \tan\phi$ and $y = \mathrm{cosec}\,\phi + \cot\phi$ then show that $xy + x - y + 1 = 0$
[**Hint:** Find $xy + 1$ and then show that $x - y = -(xy + 1)$]

25. If θ lies in the first quadrant and $\cos\theta = \frac{8}{17}$, then find the value of

$\cos(30° + \theta) + \cos(45° - \theta) + \cos(120° - \theta)$.

26. Find the value of the expression $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$

[**Hint:** Simplify the expression to $2\left(\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8}\right)$

$$= 2\left[\left(\cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8}\right)^2 - 2\cos^2\frac{\pi}{8}\cos^2\frac{3\pi}{8}\right]$$

27. Find the general solution of the equation $5\cos^2\theta + 7\sin^2\theta - 6 = 0$

28. Find the general solution of the equation
$\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

29. Find the general solution of the equation $(\sqrt{3} - 1)\cos\theta + (\sqrt{3} + 1)\sin\theta = 2$

[**Hint**: Put $\sqrt{3} - 1 = r\sin\alpha$, $\sqrt{3} + 1 = r\cos\alpha$ which gives $\tan\alpha = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$
$\alpha = \frac{\pi}{12}$]

Objective Type Questions

Choose the correct answer from the given four options in the Exercises 30 to 59 (M.C.Q.).

30. If $\sin\theta + \operatorname{cosec}\theta = 2$, then $\sin^2\theta + \operatorname{cosec}^2\theta$ is equal to

(A) 1 (B) 4
(C) 2 (D) None of these

31. If $f(x) = \cos^2 x + \sec^2 x$, then

(A) $f(x) < 1$ (B) $f(x) = 1$
(C) $2 < f(x) < 1$ (D) $f(x) \geq 2$

[**Hint**: A.M ≥ G.M.]

32. If $\tan\theta = \frac{1}{2}$ and $\tan\phi = \frac{1}{3}$, then the value of $\theta + \phi$ is

(A) $\frac{\pi}{6}$ (B) π (C) 0 (D) $\frac{\pi}{4}$

33. Which of the following is not correct?

(A) $\sin\theta = -\frac{1}{5}$ (B) $\cos\theta = 1$
(C) $\sec\theta = \frac{1}{2}$ (D) $\tan\theta = 20$

34. The value of $\tan 1° \tan 2° \tan 3° \ldots \tan 89°$ is

(A) 0 (B) 1
(C) $\frac{1}{2}$ (D) Not defined

35. The value of $\frac{1-\tan^2 15°}{1+\tan^2 15°}$ is

(A) 1 (B) $\sqrt{3}$ (C) $\frac{\sqrt{3}}{2}$ (D) 2

36. The value of cos 1° cos 2° cos 3° ... cos 179° is

(A) $\frac{1}{\sqrt{2}}$ (B) 0 (C) 1 (D) –1

37. If tan θ = 3 and θ lies in third quadrant, then the value of sin θ is

(A) $\frac{1}{\sqrt{10}}$ (B) $-\frac{1}{\sqrt{10}}$ (C) $\frac{-3}{\sqrt{10}}$ (D) $\frac{3}{\sqrt{10}}$

38. The value of tan 75° – cot 75° is equal to

(A) $2\sqrt{3}$ (B) $2+\sqrt{3}$ (C) $2-\sqrt{3}$ (D) 1

39. Which of the following is correct?

(A) $\sin 1° > \sin 1$ (B) $\sin 1° < \sin 1$

(C) $\sin 1° = \sin 1$ (D) $\sin 1° = \frac{\pi}{18°} \sin 1$

[**Hint**: 1 radian $= \frac{180°}{\pi} = 57° 30'$ approx]

40. If $\tan \alpha = \frac{m}{m+1}$, $\tan \beta = \frac{1}{2m+1}$, then α + β is equal to

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{4}$

41. The minimum value of 3 cos$x$ + 4 sin$x$ + 8 is

(A) 5 (B) 9 (C) 7 (D) 3

42. The value of tan 3A – tan 2A – tan A is equal to

(A) tan 3A tan 2A tan A

(B) – tan 3A tan 2A tan A

(C) tan A tan 2A – tan 2A tan 3A – tan 3A tan A

(D) None of these

43. The value of sin (45° + θ) – cos (45° – θ) is

(A) 2 cosθ (B) 2 sinθ (C) 1 (D) 0

44. The value of $\cot\left(\frac{\pi}{4}+\theta\right)\cot\left(\frac{\pi}{4}-\theta\right)$ is

(A) –1 (B) 0 (C) 1 (D) Not defined

45. cos 2θ cos 2ϕ + sin² (θ – ϕ) – sin² (θ + ϕ) is equal to

(A) sin 2(θ + ϕ) (B) cos 2(θ + ϕ)

(C) sin 2(θ – ϕ) (D) cos 2(θ – ϕ)

[**Hint**: Use sin² A – sin² B = sin (A + B) sin (A – B)]

46. The value of cos 12° + cos 84° + cos 156° + cos 132° is

(A) $\frac{1}{2}$ (B) 1 (C) $-\frac{1}{2}$ (D) $\frac{1}{8}$

47. If tan A = $\frac{1}{2}$, tan B = $\frac{1}{3}$, then tan (2A + B) is equal to

(A) 1 (B) 2 (C) 3 (D) 4

48. The value of $\sin\frac{\pi}{10}\sin\frac{13\pi}{10}$ is

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) $-\frac{1}{4}$ (D) 1

[**Hint**: Use sin 18° = $\frac{\sqrt{5}-1}{4}$ and cos 36° = $\frac{\sqrt{5}+1}{4}$]

49. The value of sin 50° – sin 70° + sin 10° is equal to

(A) 1 (B) 0 (C) $\frac{1}{2}$ (D) 2

50. If sin θ + cos θ = 1, then the value of sin 2θ is equal to

(A) 1 (B) $\frac{1}{2}$ (C) 0 (D) –1

51. If $\alpha + \beta = \frac{\pi}{4}$, then the value of $(1 + \tan \alpha)(1 + \tan \beta)$ is

(A) 1 (B) 2

(C) – 2 (D) Not defined

52. If $\sin \theta = \frac{-4}{5}$ and θ lies in third quadrant then the value of $\cos \frac{\theta}{2}$ is

(A) $\frac{1}{5}$ (B) $-\frac{1}{\sqrt{10}}$ (C) $-\frac{1}{\sqrt{5}}$ (D) $\frac{1}{\sqrt{10}}$

53. Number of solutions of the equation $\tan x + \sec x = 2 \cos x$ lying in the interval $[0, 2\pi]$ is

(A) 0 (B) 1 (C) 2 (D) 3

54. The value of $\sin\frac{\pi}{18} + \sin\frac{\pi}{9} + \sin\frac{2\pi}{9} + \sin\frac{5\pi}{18}$ is given by

(A) $\sin\frac{7\pi}{18} + \sin\frac{4\pi}{9}$ (B) 1

(C) $\cos\frac{\pi}{6} + \cos\frac{3\pi}{7}$ (D) $\cos\frac{\pi}{9} + \sin\frac{\pi}{9}$

55. If A lies in the second quadrant and 3 tan A + 4 = 0, then the value of 2 cotA – 5 cos A + sin A is equal to

(A) $\frac{-53}{10}$ (B) $\frac{23}{10}$ (C) $\frac{37}{10}$ (D) $\frac{7}{10}$

56. The value of $\cos^2 48° - \sin^2 12°$ is

(A) $\frac{\sqrt{5}+1}{8}$ (B) $\frac{\sqrt{5}-1}{8}$

(C) $\frac{\sqrt{5}+1}{5}$ (D) $\frac{\sqrt{5}+1}{2\sqrt{2}}$

[**Hint**: Use $\cos^2 A - \sin^2 B = \cos (A + B) \cos (A - B)$]

57. If $\tan \alpha = \frac{1}{7}$, $\tan \beta = \frac{1}{3}$, then $\cos 2\alpha$ is equal to

(A) $\sin 2\beta$ (B) $\sin 4\beta$ (C) $\sin 3\beta$ (D) $\cos 2\beta$

58. If $\tan \theta = \frac{a}{b}$, then $b \cos 2\theta + a \sin 2\theta$ is equal to

(A) a (B) b (C) $\frac{a}{b}$ (D) None

59. If for real values of x, $\cos \theta = x + \frac{1}{x}$, then

(A) θ is an acute angle (B) θ is right angle

(C) θ is an obtuse angle (D) No value of θ is possible

Fill in the blanks in Exercises 60 to 67 :

60. The value of $\frac{\sin 50°}{\sin 130°}$ is _______.

61. If $k = \sin\left(\frac{\pi}{18}\right)\sin\left(\frac{5\pi}{18}\right)\sin\left(\frac{7\pi}{18}\right)$, then the numerical value of k is _______.

62. If $\tan A = \frac{1-\cos B}{\sin B}$, then $\tan 2A$ = _______.

63. If $\sin x + \cos x = a$, then

(i) $\sin^6 x + \cos^6 x$ = _______

(ii) $|\sin x - \cos x|$ = _______.

64. In a triangle ABC with $\angle C = 90°$ the equation whose roots are tan A and tan B is _______.

[**Hint:** $A + B = 90° \Rightarrow \tan A \tan B = 1$ and $\tan A + \tan B = \frac{2}{\sin 2A}$]

65. $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$ = _______.

66. Given $x > 0$, the values of $f(x) = -3\cos\sqrt{3+x+x^2}$ lie in the interval _______.

67. The maximum distance of a point on the graph of the function $y = \sqrt{3}\sin x + \cos x$ from x-axis is ________.

In each of the Exercises 68 to 75, state whether the statements is True or False? Also give justification.

68. If $\tan A = \dfrac{1-\cos B}{\sin B}$, then $\tan 2A = \tan B$

69. The equality $\sin A + \sin 2A + \sin 3A = 3$ holds for some real value of A.

70. $\sin 10°$ is greater than $\cos 10°$.

71. $\cos\dfrac{2\pi}{15}\cos\dfrac{4\pi}{15}\cos\dfrac{8\pi}{15}\cos\dfrac{16\pi}{15} = \dfrac{1}{16}$

72. One value of θ which satisfies the equation $\sin^4\theta - 2\sin^2\theta - 1$ lies between 0 and 2π.

73. If $\operatorname{cosec} x = 1 + \cot x$ then $x = 2n\pi,\ 2n\pi + \dfrac{\pi}{2}$

74. If $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$, then $\theta = \dfrac{n\pi}{3} + \dfrac{\pi}{9}$

75. If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, then $\cos\left(\theta - \dfrac{\pi}{4}\right) = \pm\dfrac{1}{2\sqrt{2}}$

76. In the following match each item given under the column C_1 to its correct answer given under the column C_2:

(a) $\sin(x+y)\sin(x-y)$	(i) $\cos^2 x - \sin^2 y$
(b) $\cos(x+y)\cos(x-y)$	(ii) $\dfrac{1-\tan\theta}{1+\tan\theta}$
(c) $\cot\left(\dfrac{\pi}{4}+\theta\right)$	(iii) $\dfrac{1+\tan\theta}{1-\tan\theta}$
(d) $\tan\left(\dfrac{\pi}{4}+\theta\right)$	(iv) $\sin^2 x - \sin^2 y$