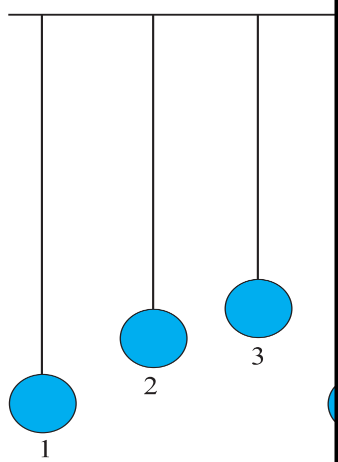
swings is a good example of resonance. You might have realised that the skill in swinging to greater heights lies in the synchronisation of the rhythm of pushing against the ground with the natural frequency of the swing.

To illustrate this point further, let us consider a set of five simple pendulums of assorted lengths suspended from a common rope as shown in Fig. 14.22. The pendulums 1 and 4 have the same lengths and the others have different lengths. Now, let us set pendulum 1 into motion. The energy from this pendulum gets transferred to other pendulums through the connecting rope and they start oscillating. The driving force is provided through the connecting rope. The frequency of this force is the frequency with which pendulum 1 oscillates. If we observe the response of pendulums 2, 3 and 5, they first start oscillating with their n[redacted] of oscillations and different a[redacted]



[redacted]

Fig. 14.22 *Five simple pen*[redacted] *lengths suspended from a common support.*

motion is gradually damped and not sustained. Their frequencies of oscillation gradually change, and ultimately, they oscillate with the frequency of pendulum 1, i.e., the frequency of the driving force but with different amplitudes. They oscillate with small amplitudes. The response of pendulum 4 is in contrast to this set of pendulums. It oscillates with the same frequency as that of pendulum 1 and its amplitude gradually picks up and becomes very large. A resonance-like response is seen. This happens because in this the condition for resonance is satisfied, i.e. the natural frequency of the system coincides with that of the driving force.

We have so far considered oscillating systems which have just one natural frequency. In general, a system may have several natural [redacted]ee examples of such [redacted]s, air columns, etc.) in [redacted]chanical structure, like [redacted] an aircraft may have [redacted]ral frequencies. An [redacted]r disturbance will set [redacted]llation. If, accidentally, [redacted]happens to be close to [redacted]encies of the system, [redacted]lation will shoot up [redacted] possible damage. This [redacted]f step while crossing a [redacted]on, an earthquake will [redacted]age to all buildings in [redacted]they are built with the [redacted]aterial. The natural [redacted] depend on its height, [redacted], and the nature of [redacted] one with its natural frequency close to the frequency of seismic wave is likely to be damaged more.

SUMMARY

1. The motion that repeats itself is called *periodic motion.*
2. The *period T* is the time required for one complete oscillation, or cycle. It is related to the frequency ν by,

$$T = \frac{1}{\nu}$$

The *frequency* ν of periodic or oscillatory motion is the number of oscillations per unit time. In the SI, it is measured in hertz :

1 hertz = 1 Hz = 1 oscillation per second = $1 s^{-1}$

3. In *simple harmonic motion* (SHM), the displacement $x(t)$ of a particle from its equilibrium position is given by,

$$x(t) = A \cos(\omega t + \phi) \quad \text{(displacement)},$$

in which A is the *amplitude* of the displacement, the quantity $(\omega t + \phi)$ is the phase of the motion, and ϕ is the *phase constant*. The *angular frequency* ω is related to the period and frequency of the motion by,

$$\omega = \frac{2\pi}{T} = 2\pi\nu \quad \text{(angular frequency)}.$$

4. Simple harmonic motion can also be viewed as the projection of uniform circular motion on the diameter of the circle in which the latter motion occurs.
5. The particle velocity and acceleration during SHM as functions of time are given by,

$$v(t) = -\omega A \sin(\omega t + \phi) \quad \text{(velocity)},$$

$$a(t) = -\omega^2 A \cos(\omega t + \phi)$$

$$= -\omega^2 x(t) \quad \text{(acceleration)},$$

Thus we see that both velocity and acceleration of a body executing simple harmonic motion are periodic functions, having the velocity *amplitude* $v_m = \omega A$ and *acceleration amplitude* $a_m = \omega^2 A$, respectively.
6. The force acting in a simple harmonic motion is proportional to the displacement and is always directed towards the centre of motion.
7. A particle executing simple harmonic motion has, at any time, kinetic energy $K = ½ mv^2$ and potential energy $U = ½ kx^2$. If no friction is present the mechanical energy of the system, $E = K + U$ always remains constant even though K and U change with time.
8. A particle of mass m oscillating under the influence of Hooke's law restoring force given by $F = -k x$ exhibits simple harmonic motion with

$$\omega = \sqrt{\frac{k}{m}} \quad \text{(angular frequency)}$$

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{(period)}$$

Such a system is also called a linear oscillator.
9. The motion of a simple pendulum swinging through small angles is approximately simple harmonic. The period of oscillation is given by,

$$T = 2\pi\sqrt{\frac{L}{g}}$$

10. The mechanical energy in a real oscillating system decreases during oscillations because external forces, such as drag, inhibit the oscillations and transfer mechanical energy to thermal energy. The real oscillator and its motion are then said to be *damped*. If the

damping force is given by $F_d = -bv$, where v is the velocity of the oscillator and b is a *damping constant*, then the displacement of the oscillator is given by,

$$x(t) = A\,e^{-bt/2m}\cos(\omega' t + \phi)$$

where ω', the angular frequency of the damped oscillator, is given by

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

If the damping constant is small then $\omega' \approx \omega$, where ω is the angular frequency of the undamped oscillator. The mechanical energy E of the damped oscillator is given by

$$E(t) = \frac{1}{2}kA^2 e^{-bt/m}$$

11. If an external force with angular frequency ω_d acts on an oscillating system with natural angular frequency ω, the system oscillates with angular frequency ω_d. The amplitude of oscillations is the greatest when

$$\omega_d = \omega$$

a condition called *resonance*.

Physical quantity	Symbol	Dimensions	Unit	Remarks
Period	T	[T]	s	The least time for motion to repeat itself
Frequency	ν (or f)	$[T^{-1}]$	s^{-1}	$\nu = \frac{1}{T}$
Angular frequency	ω	$[T^{-1}]$	s^{-1}	$\omega = 2\pi\nu$
Phase constant	ϕ	Dimensionless	rad	Initial value of phase of displacement in SHM
Force constant	k	$[MT^{-2}]$	$N\ m^{-1}$	Simple harmonic motion $F = -k\,x$

POINTS TO PONDER

1. The period T is the *least time* after which motion repeats itself. Thus, motion repeats itself after nT where n is an integer.
2. Every periodic motion is not simple harmonic motion. Only that periodic motion governed by the force law $F = -k\,x$ is simple harmonic.
3. Circular motion can arise due to an inverse-square law force (as in planetary motion) as well as due to simple harmonic force in two dimensions equal to: $-m\omega^2 r$. In the latter case, the phases of motion, in two perpendicular directions (x and y) must differ by $\pi/2$. Thus, for example, a particle subject to a force $-m\omega^2 r$ with initial position (0, A) and velocity (ωA, 0) will move uniformly in a circle of radius A.
4. For linear simple harmonic motion with a given ω, two initial conditions are necessary and sufficient to determine the motion completely. The initial conditions may be (i) initial position and initial velocity or (ii) amplitude and phase or (iii) energy and phase.

5. From point 4 above, given amplitude or energy, phase of motion is determined by the initial position or initial velocity.

6. A combination of two simple harmonic motions with arbitrary amplitudes and phases is not necessarily periodic. It is periodic only if frequency of one motion is an integral multiple of the other's frequency. However, a periodic motion can always be expressed as a sum of infinite number of harmonic motions with appropriate amplitudes.

7. The period of SHM does not depend on amplitude or energy or the phase constant. Contrast this with the periods of planetary orbits under gravitation (Kepler's third law).

8. The motion of a simple pendulum is simple harmonic for small angular displacement.

9. For motion of a particle to be simple harmonic, its displacement x must be expressible in either of the following forms :

$$x = A \cos \omega t + B \sin \omega t$$

$$x = A \cos (\omega t + \alpha), \; x = B \sin (\omega t + \beta)$$

The three forms are completely equivalent (any one can be expressed in terms of any other two forms).

Thus, damped simple harmonic motion [Eq. (14.31)] is not strictly simple harmonic. It is approximately so only for time intervals much less than $2m/b$ where b is the damping constant.

10. In forced oscillations, the steady state motion of the particle (after the forced oscillations die out) is simple harmonic motion whose frequency is the frequency of the driving frequency ω_d, not the natural frequency ω of the particle.

11. In the ideal case of zero damping, the amplitude of simple harmonic motion at resonance is infinite. Since all real systems have some damping, however small, this situation is never observed.

12. Under forced oscillation, the phase of harmonic motion of the particle differs from the phase of the driving force.

Exercises

14.1 Which of the following examples represent periodic motion?
- (a) A swimmer completing one (return) trip from one bank of a river to the other and back.
- (b) A freely suspended bar magnet displaced from its N-S direction and released.
- (c) A hydrogen molecule rotating about its centre of mass.
- (d) An arrow released from a bow.

14.2 Which of the following examples represent (nearly) simple harmonic motion and which represent periodic but not simple harmonic motion?
- (a) the rotation of earth about its axis.
- (b) motion of an oscillating mercury column in a U-tube.
- (c) motion of a ball bearing inside a smooth curved bowl, when released from a point slightly above the lower most point.
- (d) general vibrations of a polyatomic molecule about its equilibrium position.

14.3 Fig. 14.23 depicts four x-t plots for linear motion of a particle. Which of the plots represent periodic motion? What is the period of motion (in case of periodic motion) ?

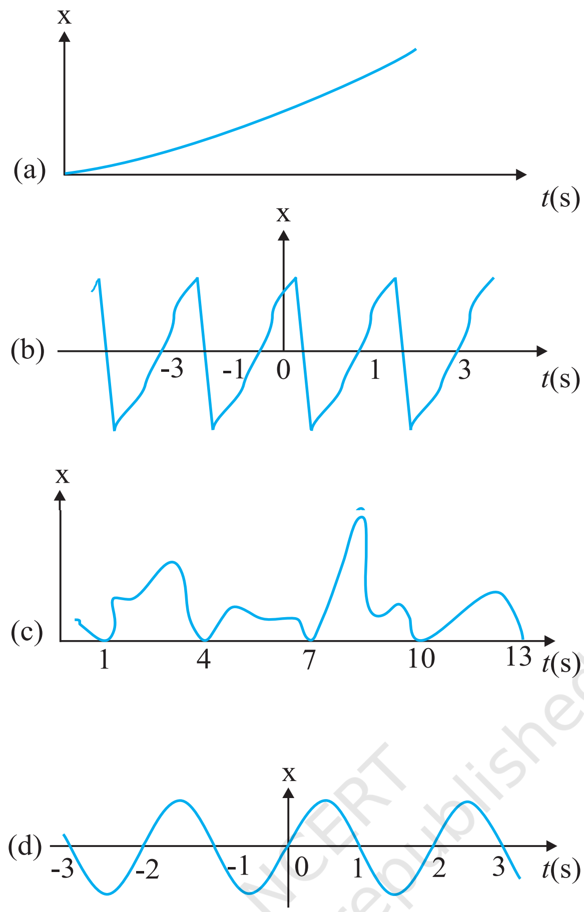


Fig. 14.23

14.4 Which of the following functions of time represent (a) simple harmonic, (b) periodic but not simple harmonic, and (c) non-periodic motion? Give period for each case of periodic motion (ω is any positive constant):

(a) $\sin \omega t - \cos \omega t$
(b) $\sin^3 \omega t$
(c) $3 \cos (\pi/4 - 2\omega t)$
(d) $\cos \omega t + \cos 3\omega t + \cos 5\omega t$
(e) $\exp (-\omega^2 t^2)$
(f) $1 + \omega t + \omega^2 t^2$

14.5 A particle is in linear simple harmonic motion between two points, A and B, 10 cm apart. Take the direction from A to B as the positive direction and give the signs of velocity, acceleration and force on the particle when it is

(a) at the end A,
(b) at the end B,
(c) at the mid-point of AB going towards A,
(d) at 2 cm away from B going towards A,
(e) at 3 cm away from A going towards B, and
(f) at 4 cm away from B going towards A.

14.6 Which of the following relationships between the acceleration a and the displacement x of a particle involve simple harmonic motion?

(a) $a = 0.7x$
(b) $a = -200x^2$
(c) $a = -10x$
(d) $a = 100x^3$

14.7 The motion of a particle executing simple harmonic motion is described by the displacement function,

$x(t) = A \cos (\omega t + \phi)$.

If the initial (t = 0) position of the particle is 1 cm and its initial velocity is ω cm/s, what are its amplitude and initial phase angle ? The angular frequency of the particle is π s^{-1}. If instead of the cosine function, we choose the sine function to describe the SHM : $x = B \sin (\omega t + \alpha)$, what are the amplitude and initial phase of the particle with the above initial conditions.

14.8 A spring balance has a scale that reads from 0 to 50 kg. The length of the scale is 20 cm. A body suspended from this balance, when displaced and released, oscillates with a period of 0.6 s. What is the weight of the body ?

14.9 A spring having with a spring constant 1200 N m^{-1} is mounted on a horizontal table as shown in Fig. 14.24. A mass of 3 kg is attached to the free end of the spring. The mass is then pulled sideways to a distance of 2.0 cm and released.

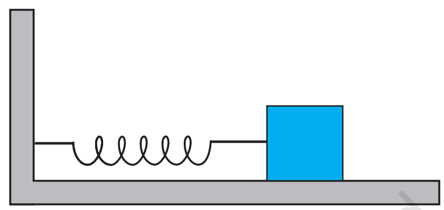


Fig. 14.24

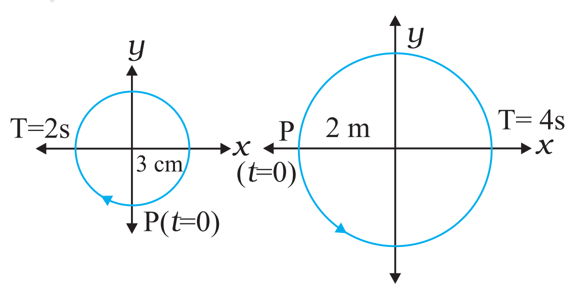
Determine (i) the frequency of oscillations, (ii) maximum acceleration of the mass, and (iii) the maximum speed of the mass.

14.10 In Exercise 14.9, let us take the position of mass when the spring is unstreched as x = 0, and the direction from left to right as the positive direction of x-axis. Give x as a function of time t for the oscillating mass if at the moment we start the stopwatch (t = 0), the mass is

(a) at the mean position,
(b) at the maximum stretched position, and
(c) at the maximum compressed position.

In what way do these functions for SHM differ from each other, in frequency, in amplitude or the initial phase?

14.11 Figures 14.25 correspond to two circular motions. The radius of the circle, the period of revolution, the initial position, and the sense of revolution (i.e. clockwise or anti-clockwise) are indicated on each figure.



(a) (b)

Fig. 14.25

Obtain the corresponding simple harmonic motions of the x-projection of the radius vector of the revolving particle P, in each case.

14.12 Plot the corresponding reference circle for each of the following simple harmonic motions. Indicate the initial (t =0) position of the particle, the radius of the circle, and the angular speed of the rotating particle. For simplicity, the sense of rotation may be fixed to be anticlockwise in every case: (x is in cm and t is in s).

(a) $x = -2 \sin (3t + \pi/3)$

(b) $x = \cos (\pi/6 - t)$

(c) $x = 3 \sin (2\pi t + \pi/4)$

(d) $x = 2 \cos \pi t$

14.13 Figure 14.26 (a) shows a spring of force constant k clamped rigidly at one end and a mass m attached to its free end. A force **F** applied at the free end stretches the spring. Figure 14.26 (b) shows the same spring with both ends free and attached to a mass m at either end. Each end of the spring in Fig. 14.26(b) is stretched by the same force **F**.

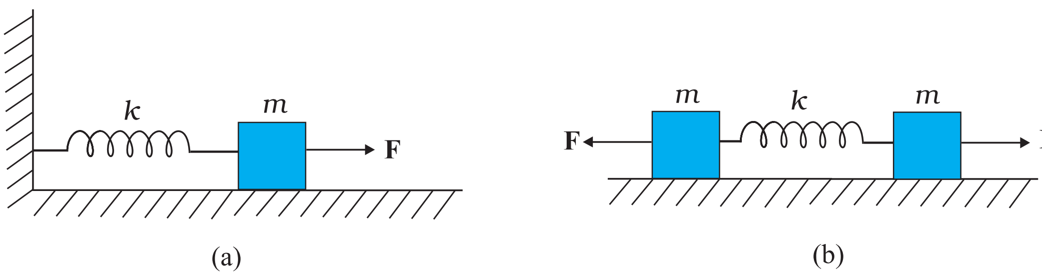


Fig. 14.26

(a) What is the maximum extension of the spring in the two cases ?
(b) If the mass in Fig. (a) and the two masses in Fig. (b) are released, what is the period of oscillation in each case ?

14.14 The piston in the cylinder head of a locomotive has a stroke (twice the amplitude) of 1.0 m. If the piston moves with simple harmonic motion with an angular frequency of 200 rad/min, what is its maximum speed ?

14.15 The acceleration due to gravity on the surface of moon is 1.7 m s^{-2}. What is the time period of a simple pendulum on the surface of moon if its time period on the surface of earth is 3.5 s ? (g on the surface of earth is 9.8 m s^{-2})

14.16 Answer the following questions :
(a) Time period of a particle in SHM depends on the force constant k and mass m of the particle:

$T = 2\pi\sqrt{\frac{m}{k}}$. A simple pendulum executes SHM approximately. Why then is the time period of a pendulum independent of the mass of the pendulum?

(b) The motion of a simple pendulum is approximately simple harmonic for small angle oscillations. For larger angles of oscillation, a more involved analysis shows that T is greater than $2\pi\sqrt{\frac{l}{g}}$. Think of a qualitative argument to appreciate this result.

(c) A man with a wristwatch on his hand falls from the top of a tower. Does the watch give correct time during the free fall ?

(d) What is the frequency of oscillation of a simple pendulum mounted in a cabin that is freely falling under gravity ?

14.17 A simple pendulum of length l and having a bob of mass M is suspended in a car. The car is moving on a circular track of radius R with a uniform speed v. If the pendulum makes small oscillations in a radial direction about its equilibrium position, what will be its time period ?

14.18 A cylindrical piece of cork of density of base area A and height h floats in a liquid of density ρ_l. The cork is depressed slightly and then released. Show that the cork oscillates up and down simple harmonically with a period

$$T = 2\pi \sqrt{\frac{h\rho}{\rho_l g}}$$

where ρ is the density of cork. (Ignore damping due to viscosity of the liquid).

14.19 One end of a U-tube containing mercury is connected to a suction pump and the other end to atmosphere. A small pressure difference is maintained between the two columns. Show that, when the suction pump is removed, the column of mercury in the U-tube executes simple harmonic motion.

Additional Exercises

14.20 An air chamber of volume V has a neck area of cross section a into which a ball of mass m just fits and can move up and down without any friction (Fig.14.27). Show that when the ball is pressed down a little and released , it executes SHM. Obtain an expression for the time period of oscillations assuming pressure-volume variations of air to be isothermal [see Fig. 14.27].

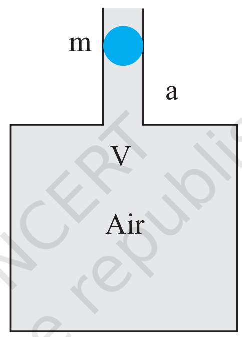


Fig.14.27

14.21 You are riding in an automobile of mass 3000 kg. Assuming that you are examining the oscillation characteristics of its suspension system. The suspension sags 15 cm when the entire automobile is placed on it. Also, the amplitude of oscillation decreases by 50% during one complete oscillation. Estimate the values of (a) the spring constant k and (b) the damping constant b for the spring and shock absorber system of one wheel, assuming that each wheel supports 750 kg.

14.22 Show that for a particle in linear SHM the average kinetic energy over a period of oscillation equals the average potential energy over the same period.

14.23 A circular disc of mass 10 kg is suspended by a wire attached to its centre. The wire is twisted by rotating the disc and released. The period of torsional oscillations is found to be 1.5 s. The radius of the disc is 15 cm. Determine the torsional spring constant of the wire. (Torsional spring constant α is defined by the relation $J = -\alpha\, \theta$, where J is the restoring couple and θ the angle of twist).

14.24 A body describes simple harmonic motion with an amplitude of 5 cm and a period of 0.2 s. Find the acceleration and velocity of the body when the displacement is (a) 5 cm (b) 3 cm (c) 0 cm.

14.25 A mass attached to a spring is free to oscillate, with angular velocity ω, in a horizontal plane without friction or damping. It is pulled to a distance x_0 and pushed towards the centre with a velocity v_0 at time $t = 0$. Determine the amplitude of the resulting oscillations in terms of the parameters ω, x_0 and v_0. [Hint : Start with the equation $x = a \cos(\omega t+\theta)$ and note that the initial velocity is negative.]